Int J Theor Phys (2009) 48: 392–402
DOI 10.1007/s10773-008-9812-0

The Characteristic Noise Connected with Continuous Measurement in Classical System without Memory

E.D. Vol

Received: 29 February 2008 / Accepted: 18 July 2008 / Published online: 22 August 2008

Abstract In the present paper we propose the way of passage from quantum theory of continuous measurements based on the Lindblad equation to its "classical" analog. The last one describes the influence of continuous measurement on the behavior of macroscopical Markov system. Such theory can be represented in the form of the Fokker-Planck equation for the distribution function of measured system. The diffusion tensor of this equation is uniquely determined by a type of the measured quantity. As the example of using of the approach proposed we describe the stationary states of linear dissipative systems induced by measurements in them. We consider possible qualitative effects connected with measurements also. In particular we demonstrate on the simple example, how in the macroscopic system, consisting of noninteracting parts, measurement of global integral of motion results in relaxation to the quasi-thermodynamic equilibrium between parts of the system. The "temperature" of such state is determined by the total energy of the system and by mean value of measured quantity.

Keywords Measurement · Markov system · Quantum-classical correspondence

PACS 03.65.Ta · 05.40.-a

1 Introduction

It is well known that the role of measurement in quantum mechanics is much broader than in the classical physics, where it's role only passive and consists in obtaining by an experimenter the necessary information about the observed system. It is essential to emphasize that all information about the macroscopic system may be obtained without disturbance of it's state by the measuring device(meter). Conversely in the quantum mechanics according to the uncertainty principle it is impossible to eliminate back reaction of the meter on the state

E.D. Vol (✉)
B. Verkin Institute for Low Temperature Physics and Engineering
of National Academy of Sciences of Ukraine, 47 Lenin Ave., 61103 Kharkov, Ukraine
e-mail: vol@ilt.kharkov.ua

of measured system. Nevertheless using the quantum theory of continuous measurements (see the review [1] and references in it), it is possible not only to calculate the influence of measurement on a state of a system but also to use this influence for state monitoring. On the other hand since there is no impenetrable border between classical and quantum world, the natural question emerges: whether it is possible on the basis of classical concepts to evaluate the influence of a meter on a state of measured macro- or mesoscopical system whatever small this influence would be? The main goal of our paper is exactly to propose the feasible approach to answer this question. The paper is organized as follows. In the Sect. 2 the simplest (minimal) quantum model of continuous measurement process is considered. We present that in the limit $\hbar \to 0$ this model may be formulated using Fokker-Planck equation for distribution function of corresponding classical system. The diffusion tensor of this equation is uniquely determined by measured physical quantity. In the Sect. 3 the stationary states of linear classical systems induced by the measuring process in them are considered. We demonstrate that in this situation the "thermodynamics" of measuring process may be developed. This one has some features which make it different from usual linear thermodynamics. In the Sect. 4 some qualitative effects caused by the influence of measuring process on the evolution of macro- or mesoscopical system are considered. In particular, as the example of such phenomenon the thermalization effect between noninteracting parts of composite classic system is investigated. This effect is stimulated by measuring of the global integral of motion of the corresponding system.

2 Derivation of the Fokker-Planck Equation Describing the Measurement Process for Macroscopical System

As one knows, the description of evolution of an open quantum system Q in Markov approximation is given by Lindblad equation for density matrix $\hat{\rho}$ of this system [2]:

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] + \sum_i ([\hat{R}_i\hat{\rho}, \hat{R}_i^\dagger] + [\hat{R}_i, \hat{\rho}\hat{R}_i^\dagger]), \tag{2.1}$$

where $\hat{H} = \hat{H}^\dagger$ and $\hat{R}_i$, $\hat{R}_i^\dagger$ are set of operators describing both internal dynamics of system Q and its interaction with an environment.

On the other hand, the description of behavior of classical open system C effected by additional stochastic forces (noise) can be obtained in Markov approximation in the framework of the Fokker-Planck equation for its distribution function $f(q,t)$ [3]:

$$\frac{df}{dt} = -\frac{\partial}{\partial q_i}(K_i f) + \frac{\partial^2}{\partial q_i \partial q_k}(D_{ik} f), \tag{2.2}$$

where: $q \equiv \{q_i\}$—set of the coordinates describing a state of a system C, $K_i(q)$—drift velocity of C, determined by the equations of motion: $dq_i/dt = K_i(q)$, and $D_{ik}(q)$—diffusion tensor whose form is defined by the correlation tensor of stochastic forces field. Assume now that quantum system Q has classical analog C_Q. It looks very plausible that between these two descriptions close connection may exist. Such connection should allow one starting from a set of operators $\hat{H}$, $\hat{R}_i$, $\hat{R}_i^\dagger$ for Q, in the limit $\hbar \to 0$, to determine the expressions for $K_i(q)$ and $D_{ik}(q)$ of the classical system C_Q by the regular procedure. The correspondence between classical and quantum description for open Markov system was analyzed in the paper of the author [4]. It has been shown, that in the first order on $\hbar$ Lindblad equation

(2.1) for ρ can be reduced to the classical Liouville equation for $f(q,t)$, i.e. actually to (2.2) but without diffusive term. Meanwhile in [4] author did not take into account possible measurements performed under Q (because their effect in the first order on $\hbar$ is strictly equal to zero). In the present paper we want to show that taking into account the influence of measurements on C_Q evolution results in the second order on $\hbar$ to the Fokker-Planck equation for $f(q,t)$ with diffusion tensor whose form is uniquely determined by measured quantity. Let us demonstrate this statement at the example of the open system Q_1 with one degree of freedom whose evolution is prescribed by measurement of physical quantity (observable) O. According to the quantum mechanics, the hermitian operator $\hat{O}$ corresponds to the observable O. The equation for evolution of density matrix ρ of system Q_1 under the continuous measurement of O is (see Ref. [1]):

$$\frac{d\hat{\rho}}{dt} = \frac{\gamma}{2}[\hat{O}\hat{\rho}, \hat{O}] + \frac{\gamma}{2}[\hat{O}, \hat{\rho}\hat{O}] = -\frac{\gamma}{2}[\hat{O}, [\hat{O}, \hat{\rho}]], \tag{2.3}$$

where γ is coupling constant between the meter and the measured system. Note that in this paper we restrict ourselves by investigating of the minimal measuring model describing only the decoherence of measured system. As for the dissipation effects they must be investigated in the framework of more general non-minimal disturbance model [5], but we can neglect them for our purposes. The point is that the dissipation effects in the macroscopical system, being the result of nonconservative forces acting on the system, may be precisely taken into account in the equations of motion for the considered system. Therefore the dissipation affects on the drift term in the Fokker-Planck equation for the distribution function only, but not affects on the type of diffusion tensor which we are interested in here.

Following to the method of [4], we make passage to the limit $\hbar \to 0$. Under such passage one must replace the density matrix ρ of Q_1 by it's classical counterpart—distribution function $f(q,p,t)$ of system C_{Q_1}, and commutators in the r.h.s. of (2.3) by Poisson brackets according to the Dirac rule:

$$[\hat{A}, \hat{B}] \to i\hbar\{\hat{A}, \hat{B}\},$$

where

$$\{\hat{A}, \hat{B}\} = \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q}$$

and $A(q,p)$, $B(q,p)$ are classical analogs of operators $\hat{A}$ and $\hat{B}$. Calculating double commutator according to this rule in the leading order on $\hbar$ we easily come to the desired equation for distribution function $f(q,p,t)$ of C_{Q_1}:

$$\frac{\partial f}{\partial t} = \frac{\gamma\hbar^2}{2}\{O\{O,f\}\} = \frac{\gamma\hbar^2}{2}\frac{\partial}{\partial x_i}\left(D_{ik}\frac{\partial f}{\partial x_k}\right), \tag{2.4}$$

where diffusion tensor D_{ik} is determined by the measured quantity $O(q,p)$ ($O(q,p)$ is the classical counterpart of observable $\hat{O}$) with the help of relation:

$$D_{ik}(x) = \varepsilon_{il}\varepsilon_{km}\frac{\partial O}{\partial x_l}\frac{\partial O}{\partial x_m}. \tag{2.5}$$

In expressions (2.4) and (2.5) we use notation: $x_1 = q$, $x_2 = p$ and

$$\varepsilon_{ik} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

is the antisymmetric tensor of the second rank ($i, k = 1, 2$). Equation (2.4) can be easily reduced to the standard form of the Fokker- Planck equation [3]:

$$\frac{\partial f}{\partial t} = -\frac{\partial}{\partial x_i}(B_i f) + \frac{\gamma \hbar^2}{2} \frac{\partial^2}{\partial x_i \partial x_k}(D_{ik} f), \tag{2.6}$$

where

$$B_i \equiv \frac{\gamma \hbar^2}{2} \frac{\partial D_{ik}}{\partial x_k}$$

is the drift of system C_{Q_1} (caused by influence of measurement).

In view of importance of (2.4) and (2.5) for further considerations we bring the additional argument in behalf of these equations adequately describe the measurement process influence on distribution function of classical system. With this purpose let us introduce the quantity $\tilde{O}(q, p)$ such that variables $O(q, p)$ and $\tilde{O}(q, p)$ are complemented to each other. It means that their Poisson bracket $\{O, \tilde{O}\}$ is equal to one. Equation (2.4) written in variables O and $\tilde{O}$ can be represented as:

$$\frac{\partial f}{\partial t} = \frac{\partial}{\partial x_i}\left(D_{ik} \frac{\partial f}{\partial x_k}\right) \equiv \frac{\partial^2}{\partial \tilde{O}^2} f(O, \tilde{O}, t). \tag{2.7}$$

To simplify the notation we use for a moment a system of units with $\gamma \hbar^2/2 = 1$.

It is implied from (2.7) that the influence of measurement O on distribution function $f(O, \tilde{O}, t)$ expressed in variables O and $\tilde{O}$ leads to diffusion of $f(O, \tilde{O}, t)$ only on variable $\tilde{O}$. The value of distribution function for $f(O, \tilde{O}, t)$ in arbitrary time is given by well-known expression:

$$f(O, \tilde{O}, t) = \frac{1}{(2\pi t)^{1/2}} \int d\tilde{O}_1 f(O, \tilde{O}_1, t = 0) \exp\left(-\frac{(\tilde{O} - \tilde{O}_1)^2}{4t}\right). \tag{2.8}$$

It follows from the expression (2.8) that

$$f_\infty \equiv \lim_{t \to \infty} f(O, \tilde{O}, t)$$

is the function depending only on O. Therefore $f_\infty(O)$ may be regarded as distribution function for values of observable O received as a result of continuous measurement of O. In macroscopical system such interpretation completely consistent with quantum theory (see Ref. [1]) in which measurement of O results in exponential decay of non-diagonal on $\hat{O}$ matrix elements of $\hat{\rho}(t)$. It should be noted that in spite of it's plausibility the derivation of (2.4) and (2.5) outlined above may be considered only as heuristic. So, the following concrete examples showing the influence of continuous measurements on behavior of macroscopical systems help one to make more clear physical meaning of approach proposed.

3 Stationary States in the Linear Dissipative Systems Induced by Continuous Measurements

As the first example we consider the evolution of linear open system $\mathcal{L}$ under the noise induced by continuous measurement in it. For simplicity we suppose that system $\mathcal{L}$ has only one degree of freedom, and coordinates x_1 and x_2 describing its state are dimensionless. The equations of motion for variables x_1 and x_2 of such system can be written as:

$$\frac{dx_i}{dt} = A_{ik}x_k. \tag{3.1}$$

We assume that elements of matrix

$$\hat{A} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

does not depend on x_1, x_2 and t, and satisfy two additional restrictions:

(1) $\mathrm{Tr} A \equiv t_{\hat{A}} \equiv a + d < 0$,
(2) $\det A \equiv d_{\hat{A}} \equiv ad - bc > 0$.

These restrictions provide exponentional decay of solutions of (3.1) when $t \to \infty$. As established above the evolution of distribution function $f(x_1, x_2, t)$ of $\mathcal{L}$ when both drift (3.1) and continuous measurement are took into account can be described by the Fokker-Planck equation:

$$\frac{df}{dt} = -\frac{\partial}{\partial x_i}(A_{ik}x_k f) + D_{ik}\frac{\partial^2 f}{\partial x_i \partial x_k}. \tag{3.2}$$

We note that for linear system the tensor of diffusion

$$\hat{D} \equiv \begin{pmatrix} D_1 & D \\ D & D_2 \end{pmatrix}$$

corresponding to the measurement of linear x_1 and x_2 variables in $\mathcal{L}$ by virtue of (2.5) satisfies to the condition:

$$d_{\hat{D}} = D_1 D_2 - D^2 = 0. \tag{3.3}$$

Now we want to study the stationary states of $\mathcal{L}$ induced in it by process of such measurement. The method used for this purpose as a matter of fact is the same which used in the statistical physics under considering the fluctuations of physical quantities near equilibrium state (see e.g. Ref. [6]). We are looking for stationary solutions of the Fokker-Planck equation in a standard form (3.2): $f(x_1, x_2) \sim \exp(S)$, where $S(x_1, x_2) = -\beta_{ik}x_i x_k/2$ is negative definite quadratic form of x_1 and x_2 which plays a role of entropy for stationary state. Substituting this expression for $f(x_1, x_2)$ into (3.2), and, equating the coefficients at identical powers of variables x_1 and x_2, we obtain two equations for unknown symmetric matrix $\hat{\beta} \equiv \beta_{ik}$:

$$\mathrm{Tr}\hat{A} = -\mathrm{Tr}\hat{D}\hat{\beta}, \tag{3.4}$$

$$-\frac{(\hat{\beta}\hat{A} + \hat{A}^t\hat{\beta})}{2} = \hat{\beta}\hat{D}\hat{\beta}, \tag{3.5}$$

where $\hat{A}^t$ is a matrix transposed to $\hat{A}$. It is easy to see that (3.4) and (3.5) are equivalent to the single equation for a matrix $\hat{\beta}^{-1}$, reciprocal to $\hat{\beta}$:

$$\hat{A}\hat{\beta}^{-1} + \hat{\beta}^{-1}\hat{A}^t = -2\hat{D}. \tag{3.6}$$

Note, that the matrix equation (3.6) allows one to obtain solution for β^{-1} in general case when matrix $\hat{A}$ has arbitrary dimension $N \times N$ [7]. However, here we are interested in only the situation when $N = 2$. In this case the expression for $\hat{\beta}^{-1}$ satisfying to the (3.6) can be presented as:

$$\hat{\beta}^{-1} = -\frac{(t_{\hat{A}}^2 + d_{\hat{A}})}{(t_{\hat{A}} d_{\hat{A}})}\hat{D} + \frac{1}{d_{\hat{A}}}(\hat{A}\hat{D} + \hat{D}\hat{A}^t) - \frac{1}{t_{\hat{A}} d_{\hat{A}}}\hat{A}\hat{D}\hat{A}^t. \tag{3.7}$$

(We remind that $t_{\hat{A}} \equiv \mathrm{Tr}\hat{A}$ and $\hat{d}_{\hat{A}} \equiv \det \hat{A}$.) It is well known from the fluctuation theory (see Ref. [6]), that elements of a matrix $\hat{\beta}^{-1}$ coincide with the second moments of variables x_1 and x_2 in a stationary state. Therefore using (3.7) one can write down explicit expressions for these moments with the help of known elements of matrixes $\hat{A}$ and $\hat{D}$:

$$\overline{x_1^2} = \hat{\beta}_{11}^{-1} = \frac{(bc - ad - d^2)D_1 + 2bdD - b^2 D_2}{(a+d)(ad - bc)}, \tag{3.8}$$

$$\overline{x_1 x_2} = \hat{\beta}_{12}^{-1} = \frac{cdD_1 - 2adD + abD_2}{(a+d)(ad - bc)}, \tag{3.9}$$

$$\overline{x_2^2} = \hat{\beta}_{22}^{-1} = \frac{-c^2 D_1 + 2acD + (bc - a^2 - ad)D_2}{(a+d)(ad - bc)}. \tag{3.10}$$

Now having in hands expression for $\hat{\beta}_{ik}^{-1}$ and hence for the entropy $S(x_1, x_2)$ we can to construct "thermodynamics" of measurement process for linear open systems. By analogy to usual linear thermodynamics we define "forces" X_i as:

$$X_i = \frac{\partial S}{\partial x_i} = -\beta_{ik} x_k. \tag{3.11}$$

Coordinates x_1 and x_2 describing system state are expressed by forces X_i as:

$$x_i = -\beta_{ik}^{-1} X_k. \tag{3.12}$$

It is convenient to introduce the kinetic matrix $\hat{L} = -\hat{A}\hat{\beta}^{-1}$ and by means of it to write down the equations of motion for $\mathcal{L}$ (3.1) in the standard form as the connection between "flows" $j_i \equiv dx_i/dt$ and "forces" X_i:

$$j_i = A_{ik} x_k = -(\hat{A}\hat{\beta}^{-1})_{ik} X_k = L_{ik} X_k. \tag{3.13}$$

Comparing expression for $\hat{L}$ with (3.6), we obtain the relation:

$$\hat{L} + \hat{L}^\dagger = 2\hat{D}. \tag{3.14}$$

It is known that in linear nonequilibrium thermodynamics the kinetic matrix $\hat{L}$ is symmetric (if magnetic field is absent), i.e. $\hat{L} = \hat{L}^t$. This fundamental result for the first time obtained by Onsager [8] follows from the symmetry of equations of motion with respect to time inversion. In the case of arbitrary open linear system $\mathcal{L}$ the equations of motion (3.1) obviously do not possess such symmetry. It is interesting to note that under definite restrictions on a measured quantity kinetic matrix $\hat{L}$ turns out to be symmetric. Let us find these conditions in explicit form. For this purpose we substitute expression for β^{-1} from (3.7) into definition of kinetic matrix $\hat{L} = -\hat{A}\hat{\beta}^{-1}$ and after simple algebra obtain the following relation:

$$\hat{L} = \hat{D} + \frac{1}{t_{\hat{A}}}(\hat{A}\hat{D} - \hat{D}\hat{A}^t). \tag{3.15}$$

From (3.15) follows that the matrix $\hat{L}$ becomes symmetric under condition: $\hat{A}\hat{D} = \hat{D}\hat{A}^t$ or when elements of matrixes $\hat{A}$ and $\hat{D}$ are connected as:

$$bD_2 - cD_1 + (a-d)D = 0. \tag{3.16}$$

Recalling now the general restriction (3.3) on the elements $\hat{D}$ which corresponds to the measurement process we come as a result to the following conclusion. For arbitrary open linear system an observable does exist, whose continuous measurement induces the stationary state of a system with symmetric matrix $\hat{L}$. It is worth to note that the same measurement results in the maximal correlation between coordinates. Let us prove this statement. We introduce the coefficient of correlation η_c between x_1 and x_2 by means of standard definition:

$$\eta_c = \frac{\overline{x_1 x_2}}{(\overline{x_1^2}\,\overline{x_2^2})^{1/2}}. \tag{3.17}$$

It is implied in (3.17) that $\overline{x_1} = \overline{x_2} = 0$. Using the known expressions for second moments (3.8)–(3.10) one can easily obtain the relation:

$$\frac{1}{\eta_c^2} = \frac{\overline{x_1^2}\,\overline{x_2^2}}{(\overline{x_1 x_2})^2} = 1 + (ad - bc)\frac{[bD_2 - cD_1 + (a-d)D]^2}{(cdD_1 - 2adD + abD_2)^2}. \tag{3.18}$$

Comparing (3.18) with condition (3.16) we come to the claimed result: symmetry of kinetic matrix $\hat{L}$ leads to equality $|\eta_c| = 1$, i.e. to the maximal correlation between variables x_1 and x_2 and vice versa. The sense of the result obtained above becomes evident if we pass to variables O and $\tilde{O}$ (O—the measured quantity and $\{O, \tilde{O}\} = 1$). In these variables, directly connected with measurement, diffusion tensor has the simple form:

$$\hat{D} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

and condition of symmetry for matrix $\hat{L}$ looks as $b = 0$. The equations of motion in variables O and $\tilde{O}$ are written down as:

$$\frac{dO}{dt} = aO, \qquad \frac{d\tilde{O}}{dt} = cO + d\tilde{O}, \tag{3.19}$$

The corresponding Fokker-Planck equation for distribution function $f(O, \tilde{O})$ of stationary state may be written as:

$$\frac{\partial}{\partial O}(aOf) + \frac{\partial}{\partial \tilde{O}}[(cO + d\tilde{O})f] = \frac{\partial^2 f}{\partial \tilde{O}}. \tag{3.20}$$

As one can easily see (3.20) has normalized solution of the form:

$$f(O, \tilde{O}) = \sqrt{\frac{|d|}{2\pi}}\,\delta(O)\exp\left(-\frac{|d|\tilde{O}^2}{2}\right). \tag{3.21}$$

Thus, under the condition $\hat{A}\hat{D} = \hat{D}\hat{A}^t$, $f(O, \tilde{O})$ turns out to be proportional to delta-function of measured quantity. It means the "freezing" of the observed quantity. Such case automatically results in the maximal correlation between coordinates of $\mathcal{L}$ and to the symmetry of kinetic matrix $\hat{L}$. One can say that under condition (3.16) we have some analog of quantum Zeno effect (see Ref. [9]) for classical system.

Now let us discuss briefly the possibility of experimental observation the effects connected with influence of measurement on behaviour of macro- or mesoscopical systems. Two main obstacles can hinder such observation: (1) the smallness of the measurement noise proportional, as we saw, to $\gamma\hbar^2$ and (2) the unavoidable presence at experiment of extraneous noise of different nature (thermal, shot and so on), which can suppress effects connected with measurement. The first obstacle is essential mainly for linear systems. Really, it is well-known, that in nonlinear systems not far from bifurcation point even weak external noise can result in qualitative modification of the system behaviour (see e.g. Ref. [10]). The simple example of such bifurcation under influence of measuring noise is considered in [11].

Here we want to make the next remark. In the number of cases the coupling constant of the meter with the measured system—γ is proportional to $1/\hbar^2$. In such situations the Fokker-Planck equation following from the Lindblad equation in the limit $\hbar \to 0$ possesses the explicit statistical interpretation. For example, in the model of quantum diffusion considered in [12] (further we use the simplified version of [12] outlined in [1]) the measuring procedure of the position r of the particle moving through the crystal and interacting with its atoms is described by the equation for density matrix of the particle $\hat{\rho}$:

$$\frac{d\hat{\rho}}{dt} = -\frac{i}{\hbar}[\hat{H}, \hat{\rho}] - \frac{\gamma}{2}[\hat{r}, [\hat{r}, \hat{\rho}]], \tag{3.22}$$

where coupling constant $\gamma = 2\eta k_B T/\hbar^2$ (k_B is the Boltzmann constant) depends on the temperature of the crystal T and friction coefficient η. According to the Sect. 2 in the limit $\hbar \to 0$ (3.22) transforms to the following equation for distribution function of the particle in the momentum space:

$$\frac{df}{dt} = \{H, f\} + D\frac{\partial^2 f}{\partial p^2}. \tag{3.23}$$

So in this case the diffusion coefficient of the particle in the momentum space—D is connected with the temperature of the medium by usual Einstein relation: $D = \eta k_B T$.

As for the second case, i.e. presence of some extraneous noise in the system, really it may be more serious obstacle for experimenter. We postpone the detailed analysis of this problem for future publications and note only that specific character of measuring noise and its selective action on various physical quantities allows one to have a chance to select it from irrelevant noise of other nature.

4 Thermolization between Noninteracting Parts Caused by the Measurement in the Composite System

In this section we consider the interesting problem connected with influence of continuous measurement in composite system on behavior of its parts. To point out the basic physical idea and conclusions following from it without complicating our account by unnecessary details we restrict our consideration by simplest example. Let us study the system C consisting of two identical noninteracting harmonic oscillators with Hamiltonian:

$$H = H_1 + H_2 = \frac{p_1^2}{2m} + \frac{kx_1^2}{2} + \frac{p_2^2}{2m} + \frac{kx_2^2}{2}. \tag{4.1}$$

The projection of angular momentum $M_z = x_1 p_2 - x_2 p_1$ is integral of motion because $\{M_z, H\} = 0$. Let us assume that continuous measurement of integral of motion M_z occurs in this system. We are interested in how such measurement will affect behavior of oscillators 1 and 2. According to the approach proposed in the Sect. 2 the evolution of distribution function of composite system may be described by the Fokker-Planck equation:

$$\begin{aligned}\frac{\partial f}{\partial t} &= -\frac{\partial}{\partial x_1}\left(\frac{\partial H}{\partial p_1} f\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial H}{\partial p_2} f\right) \\ &\quad + \frac{\partial}{\partial p_1}\left(\frac{\partial H}{\partial x_1} f\right) + \frac{\partial}{\partial p_2}\left(\frac{\partial H}{\partial x_2} f\right) + \kappa\{M_z, \{M_z, f\}\},\end{aligned} \tag{4.2}$$

where $f(\Gamma_1, \Gamma_2, t) \equiv f(x_1, p_1; x_2, p_2; t)$ is the distribution function of composite system C and $\kappa = \gamma\hbar^2/2$ is the coupling constant of the meter with measured system C.

With the help of (4.2) and using integration by parts one can obtain the dependence of mean value $\bar{A}(t)$ for any physical quantity $A(\Gamma_1, \Gamma_2, t)$ depending on time:

$$\bar{A}(t) = \int d\Gamma_1 d\Gamma_2 A(\Gamma_1, \Gamma_2, t) f(\Gamma_1, \Gamma_2, t).$$

This dependence is given by the expression:

$$\frac{dA}{dt} = \overline{\{A, H\}} + \kappa \overline{\left(p_2 \frac{\partial}{\partial p_1} - p_1 \frac{\partial}{\partial p_2} + x_2 \frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2}\right)^2 A}. \tag{4.3}$$

Using equality (4.3) one can write down the equations of motion for all second moments, i.e. for values $\overline{x_i x_k}$, $\overline{p_i p_k}$ and $\overline{x_i p_k}$ $(i = 1, 2)$ and for their linear combinations. Let us write for example the equation of motion for mean value of energy

$$E_1 = \overline{\frac{p_1^2}{2m}} + \overline{\frac{kx_1^2}{2}}$$

of oscillator 1. Using equality (4.3) we obtain:

$$\frac{dE_1}{dt} = 2\kappa(E_2 - E_1). \tag{4.4}$$

The similar equation of motion is also correct for oscillator 2:

$$\frac{dE_2}{dt} = 2\kappa(E_1 - E_2). \tag{4.5}$$

From (4.4) and (4.5) the expected result follows: energy of the composite system: $2E = E_1 + E_2$ under the measurement of M_z is conserved. Moreover one can see that equalization of subsystems energies (thermalization) takes place. We want to point out that this equalization is connected only with measurement of M_z because dynamical interaction between oscillators 1 and 2 is strictly equal to zero.

Now let us write the equations of motion for the moments $\overline{x_1 p_2}$ and $\overline{x_2 p_1}$. Using (4.3) we find:

$$\frac{d\overline{x_1 p_2}}{dt} = \frac{\overline{p_1 p_2}}{m} - k\overline{x_1 x_2} - 2\kappa(\overline{x_1 p_2 + x_2 p_1}), \tag{4.6}$$

$$\frac{d\overline{x_2 p_1}}{dt} = \frac{\overline{p_1 p_2}}{m} - k\overline{x_1 x_2} - 2\kappa(\overline{x_1 p_2 + x_2 p_1}). \tag{4.7}$$

It follows from (4.6) and (4.7) that mean value of M_z: $\overline{M_z} \equiv \overline{x_1 p_2 - x_2 p_1} \equiv M$ does not depend on time and together with total energy may be used for the characteristic of stationary state of the system during measurement M_z. The equations of motion for other second moments can be similarly obtained and the values of these moments may be determined in stationary state. Omitting trivial calculations, we present only the final results:

$$\frac{\overline{p_1^2}}{2m} = \frac{\overline{p_2^2}}{2m} = \frac{\overline{kx_1^2}}{2} = \frac{\overline{kx_2^2}}{2} = \frac{E}{2}, \tag{4.8}$$

$$\overline{x_1 p_2} = \frac{M}{2}; \qquad \overline{x_2 p_1} = -\frac{M}{2}; \tag{4.9}$$

$$\overline{x_1 x_2} = \overline{p_1 p_2} = \overline{x_1 p_1} = \overline{x_2 p_2} = 0. \tag{4.10}$$

The knowledge of all second moments allows one to write down distribution function $f_C(\Gamma_1, \Gamma_2)$ for the stationary state of composite system C in the form of Gaussian distribution, in such a way that moments $\overline{x_i x_k}$, $\overline{x_i x_k}$ and $\overline{x_i p_k}$, determined by it coincide with known (4.8), (4.9), (4.10). Let us represent $f_C(\Gamma_1, \Gamma_2)$ in a standard form $f \sim \exp(S)$, where $S(\Gamma_1, \Gamma_2) = -\beta_{\alpha\beta} y_\alpha y_\beta/2$—entropy of a stationary state of system C. We use following ordering of variables y_α ($\alpha = 1, 2, 3, 4$): $y_1 = x_1,\ y_2 = p_1,\ y_3 = x_2,\ y_4 = p_2$. The matrix $\hat{\beta}^{-1}$ reciprocal to matrix β whose elements coincide with known moments can be represented as:

$$\hat{\beta}^{-1} = \begin{pmatrix} E/k & 0 & 0 & M/2 \\ 0 & mE & -M/2 & 0 \\ 0 & -M/2 & E/k & 0 \\ M/2 & 0 & 0 & mE \end{pmatrix}. \tag{4.11}$$

In accordance with (4.11) matrix β is equal to:

$$\hat{\beta} = \begin{pmatrix} mE & 0 & 0 & -M/2 \\ 0 & E/k & M/2 & 0 \\ 0 & M/2 & mE & 0 \\ -M/2 & 0 & 0 & E/k \end{pmatrix} \left(\frac{mE^2}{k} - \frac{M^2}{4}\right)^{-1}. \tag{4.12}$$

Now with the help (4.12) one can write down the distribution function of composite system $f_C(\Gamma_1, \Gamma_2)$ in desired form:

$$f_C \sim \exp(-\beta_{eff}(H - M_z\Omega)), \tag{4.13}$$

where notations:

$$\beta_{eff} \equiv \frac{E}{\omega_0^2}\left(\frac{mE^2}{k} - \frac{M^2}{4}\right)^{-1}, \qquad \Omega \equiv \frac{M\omega_0^2}{2E}, \quad \omega_0 = \sqrt{\frac{k}{m}}$$

are used.

Representation (4.13) for distribution function $f_C(\Gamma_1,\Gamma_2)$ is the basic result of this section. The small comments are necessary to it. First of all we note that value of quantity $mE^2/k - M^2/4$ is more then zero, that is why parameter $\beta_{eff} > 0$. This statement follows from inequality $(\overline{x_1^2})(\overline{p_2^2}) \geq (\overline{x_1 p_2})^2$ and taking into account that $k\overline{x_1^2}/2 = \overline{p_2^2}/2m = E/2$, and $\overline{x_1 p_2} = M/2$. The second remark is more essential. As one can see directly from (4.13) distribution function $f_C(\Gamma_1,\Gamma_2)$ of composite system may be written down in the form of Gibbs distribution with effective Hamiltonian $H_{eff} = H - \Omega M_z$. The effective temperature of such distribution

$$k_B T_{eff} = E - \frac{M^2\omega_0^2}{4E}$$

is determined by the total energy of the system and by mean value of the measured integral of motion.

It is worth to remind that both effects: equalization of subsystems energies and relaxation to the quasi-equilibrium Gibbs distribution occur in the system of noninteracting oscillators only due to process of measurement. The observation of such effects in macro- or mesoscopical systems would be the crucial argument in behalf of approach proposed in the present paper.

5 Conclusions

In conclusion it is worth to point out the main result of the present paper. It lies in the establishment of the explicit form of connection between decoherence caused by continuous measurement in quantum Markov system and characteristic noise produced by corresponding measurement in a classical analog of such system. Besides continuous measurements at linear dissipative systems are considered in detail. Also the qualitative effects connecting with the probable influence of measuring process on the behavior of macroscopical systems were investigated. The experimental testing of effects predicted in the paper would undoubtedly cause our better understanding of the correspondence existing between classical and quantum phenomena.

Acknowledgements The author acknowledges L.A. Pastur and I.V. Krive for the discussion of the results of the paper and valuable comments.

References

1. Mensky, M.B.: Phys. Usp. **41**, 923 (1998). [Usp. Fiz. Nauk **168**, 1017 (1998)]
2. Lindblad, G.: Commun. Math. Phys. **48**, 119 (1976)
3. Risken, H.: The Fokker-Planck Equation. Springer, Berlin, New York (1984)
4. Vol, E.D.: Phys. Rev. A **73**, 062113 (2006)
5. Mensky, M.B.: Phys. Usp. **46**, 1163 (2003). [Usp. Fiz. Nauk **173**, 1199 (2003)]
6. Landau, L.D., Lifshitz, E.M.: Statistical Physics. Pergamon Press, Oxford (1978)
7. Lankaster, P.: Theory of Matrices. Academic Press, New York, London (1969)
8. Onsager, L.: Phys. Rev. **37**, 405 (1931)
9. Misra, B., Sudarshan, E.C.G.: J. Math. Phys. **18**, 756 (1977)
10. Horsthemke, W., Lefever, R.: Noise-Induced Transitions. Springer, Berlin, New York (1984)
11. Vol, E.D.: quant-phys 0706.3574 (2007)
12. Caldeira, A.O., Leggett, A.J.: Physica A **121**, 587 (1983)